we satisfy the second partial differential equation and "reduce" the first to the *transonic full-potential equation*

$$(a^2 - \phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (a^2 - \phi_y^2)\phi_{yy} = 0.$$

Although this is a complicated nonlinear second-order differential equation, it can still be classified as hyperbolic, parabolic, or elliptic. However, with nonlinear equations, the type may depend on the unknown solution. Determine the type of the transonic full-potential equation. Express your answer in terms of the Mach number

$$M^2 = \frac{u^2 + v^2}{a^2} = \frac{\phi_x^2 + \phi_y^2}{a^2}.$$

4. Show that the $\mathcal{L}^2$ norm of the solution of Burgers' equation

$$u_t + uu_x = \sigma u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$$

where σ is a positive constant, is a decreasing function of time when the initial and boundary conditions are

$$u(x,0) = \phi(x), \qquad 0 \leq x \leq 1,$$

$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

1.3 Hyperbolic Conservation Laws: Characteristics, Shock Waves, and Rankine-Hugoniot Conditions

A conservation laws states that the total amount of some quantity remains unchanged during the evolution of the solution according to the partial differential equation. In physical processes without dissipation, these quantities might be the total mass, momentum, and energy. In this introductory chapter, let us confine our attention to conservation laws in one space dimension which typically have the form

$$\boxed{\frac{d}{dt}\int_{\alpha}^{\beta} \mathbf{u}dx = -\mathbf{f}(\mathbf{u})|_{\alpha}^{\beta} = -\mathbf{f}(\mathbf{u}(\beta,t)) + \mathbf{f}(\mathbf{u}(\alpha,t)),} \tag{1.3.1}$$

where

$$\mathbf{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \\ \vdots \\ u_m(x,t) \end{bmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} f_1(\mathbf{u}) \\ f_2(\mathbf{u}) \\ \vdots \\ f_m(\mathbf{u}) \end{bmatrix}. \tag{1.3.2}$$

are density and flux vectors, respectively. Equation (1.3.1) expresses the fact that the rate of change of $\mathbf{u}$ within the "volume" $\alpha \le x \le \beta$ is equal to the change in its flux through the boundaries $x = \alpha,\ \beta$.

If $\mathbf{f}$ and $\mathbf{u}$ are smooth functions, then (1.3.1) can be written as

$$\int_\alpha^\beta [\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x]dx = 0.$$

Since this result should hold for all possible "control volumes" (α, β), the integrand must vanish, *i.e.*,

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0. \tag{1.3.3}$$

Example 1.3.1. The Euler equations for one-dimensional compressible inviscid flows may be obtained by setting all y derivatives and n to zero in equations (1.1.4) to obtain

$$\rho_t + m_x = 0, \tag{1.3.4a}$$

$$m_t + (\frac{m^2}{\rho} + p)_x = 0, \tag{1.3.4b}$$

? 2

$$e_t + [(e+p)\frac{m}{\rho}]_x = 0. \tag{1.3.4c}$$

The pressure is determined by an equation of state which we take in the rather general form $p = p(\rho, m, e)$. Recall that equations (1.3.4a), (1.3.4b), and (1.3.4c) express the facts that the mass, momentum, and energy of the fluid are neither created nor destroyed and are, hence, *conserved.* We readily see that the system (1.3.4) has the form of (1.3.3) with

$$\mathbf{u} = \begin{bmatrix} \rho \\ m \\ e \end{bmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} m \\ m^2/\rho + p \\ (e+p)m/\rho \end{bmatrix}. \tag{1.3.5}$$

Example 1.3.2. The deflection of a taut string has the form

$$u_{tt} = c^2 u_{xx}, \tag{1.3.6}$$

where $c^2 = T/\rho$ with T being the tension and ρ being the linear density of the string (cf. (1.1.2)).

Equation (1.3.6) can be written as a first-order system in a variety of ways. Perhaps the most common approach is to let

$$u_1 = u_t, \qquad u_2 = cu_x. \tag{1.3.7}$$

Differentiating with respect to t while using (1.3.6) and (1.3.7) yields

$$(u_1)_t = u_{tt} = c^2 u_{xx} = c(u_2)_x, \qquad (u_2)_t = cu_{xt} = cu_{tx} = c(u_1)_x.$$

Thus, the one-dimensional wave equation has the form of (1.3.3) with

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \mathbf{f}(\mathbf{u}) = \begin{bmatrix} -cu_2 \\ -cu_1 \end{bmatrix}.$$

For the remainder of this introductory section, we'll confine our attention to the scalar conservation law

$$\boxed{u_t + f(u)_x = 0} \tag{1.3.8}$$

and return to vector systems of conservation laws in Chapter 6. To begin, let

$$a(u) = \frac{df(u)}{du}, \tag{1.3.9a}$$

and write (1.3.8) in the *convective form*

$$u_t + a(u)u_x = 0. \tag{1.3.9b}$$

Regard $u(x,t)$ as a function of two variables and compute its total (directional) derivative

$$\boxed{du = u_t dt + u_x dx = (u_t + \frac{dx}{dt}u_x)dt.} \tag{1.3.10}$$

Let us evaluate (1.3.10) in the direction given by

$$\frac{dx}{dt} = a(u). \tag{1.3.11}$$

Then, using (1.3.9b) we see that

$$du = (u_t + au_x)dt = 0.$$

Thus, $u(x,t)$ is constant along the curve $dx/dt = a$. This curve is called a *characteristic* of the differential equation (1.3.8). Characteristics can be used to construct solutions of initial or initial boundary value problems. Let us begin with an initial value problem for (1.3.8) on $-\infty < x < \infty$, $t > 0$, with the initial condition

$$\boxed{u(x,0) = \phi(x),} \qquad -\infty < x < \infty. \tag{1.3.12}$$

Since u is constant along the characteristic curves, it must have the same value that it had initially. Thus, $u = \phi(x_0) \equiv \phi_0$ along the characteristic that passes through $(x_0, 0)$. This characteristic satisfies the ordinary initial value problem

$$\frac{dx}{dt} = a(\phi_0) \equiv a_0, \qquad t > 0, \qquad x(0) = x_0. \tag{1.3.13}$$

Integrating, we see that the characteristic is the straight line

$$x = x_0 + a_0 t. \tag{1.3.14}$$

This procedure can be repeated to trace other characteristics and thereby construct the solution. The solution of (1.3.8) is implicitly given by the formula

$$\boxed{u(x,t) = \phi(x - a(\phi_0)t);} \tag{1.3.15}$$

however, this formula is rather obscure. Let us clarify the situation with some examples.

Example 1.3.3. The simplest case occurs when a is a constant and $f(u) = au$. All of the characteristics are parallel straight lines with slope $1/a$. The solution of the initial value problem (1.3.8, 1.3.12) is $u(x,t) = \phi(x - at)$ and is, as shown in Figure 1.3.1, a wave that maintains its shape and travels with speed a.

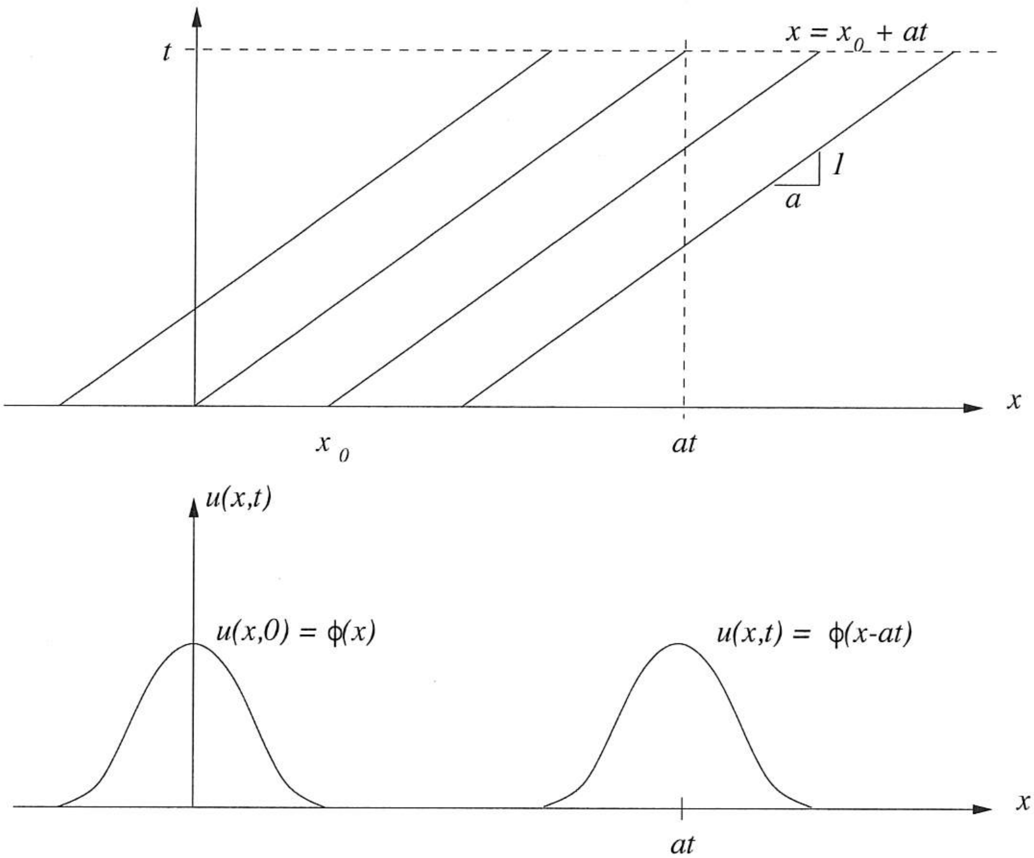
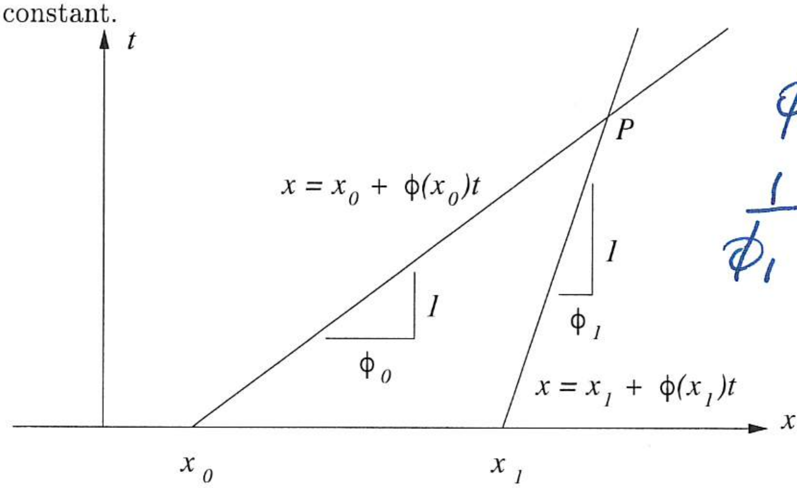


Figure 1.3.1: Characteristic curves and solution of the initial value problem (1.3.8, 1.3.12) when a is a constant.



$\phi_1 < \phi_0$

$\frac{1}{\phi_1} > \frac{1}{\phi_0}$

Figure 1.3.2: Characteristic curves for two initial points x_0 and x_1 for Burgers' equation. The characteristics intersect at a point P.

Euler's equ. in Vector form: $\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = A_c - \frac{\nabla P}{\rho}$ ×

$\vec{u} = (u, 0, 0)$

Example 1.3.4. Setting $a(u) = u$ and $\boxed{f(u) = u^2/2}$ in (1.3.8, 1.3.9a) yields the inviscid Burgers' equation

$$u_t + \underbrace{\frac{1}{2}(u^2)}_{= f(u)}{}_x = 0. \qquad \boxed{u_t + u u_x = 0} \tag{1.3.16}$$

Again, consider an initial value problem having the initial condition (1.3.12), so the characteristic is given by (1.3.14) with $a_0 = u(x_0, 0) = \phi(x_0)$, *i.e.*, In this case $a(u) = u$.

$$x = x_0 + \phi(x_0)t. \tag{1.3.17}$$

$u(x,t) = \phi(x - \phi(x_0)t)$

The characteristics are straight lines, but they are not parallel as they were in Example 1.3.3. Their slope depends on the value of the initial data; thus, the characteristic passing through the point $(x_0, 0)$ has slope $1/\phi(x_0)$. The fact that the characteristics are not parallel introduces a difficulty that was not present in the linear problem of Example 1.3.3. Consider characteristics passing through $(x_0, 0)$ and $(x_1, 0)$ and suppose that $\phi(x_0) > \phi(x_1)$ for $x_1 > x_0$. Since the slope of the characteristic passing through $(x_0, 0)$ is less than the slope of the one passing through $(x_1, 0)$, the two characteristics will intersect at a point, say, P as shown in Figure 1.3.2. The solution would appear to be multivalued at points such as P.

In order to help clarify matters, let's examine the specific choice of ϕ given by Lax [4]

$$\phi(x) = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \le x < 1 \\ 0, & \text{if } 1 \le x \end{cases}. \tag{1.3.18}$$

Using (1.3.17), we see that the characteristic passing through the point $(x_0, 0)$ satisfies

$$x = \begin{cases} x_0 + t, & \text{if } x_0 < 0 \\ x_0 + (1 - x_0)t, & \text{if } 0 \le x_0 < 1 \\ x_0, & \text{if } 1 \le x_0 \end{cases}. \tag{1.3.19}$$

Several characteristics are shown in Figure 1.3.3. The characteristics first intersect at $t = 1$. After that, the solution would presumably be multivalued, as shown in Figure 1.3.4.

It's, of course, quite possible for multivalued solutions to exist; however, (*i*) they are not observed in physical situations and (*ii*) they do not satisfy (1.3.8) in any classical sense. Discontinuous solutions are often observed in nature once characteristics of the corresponding conservation law model have intersected. They also do not satisfy (1.3.8), but they might satisfy the integral form of the conservation law (1.3.1). We examine the simplest case when two classical solutions satisfying (1.3.8) are separated by a single smooth curve $x = \xi(t)$ across which $u(x,t)$ is discontinuous. For each $t > 0$ we assume that $\alpha < \xi(t) < \beta$ and let superscripts - and + denote conditions immediately to the left and right, respectively, of $x = \xi(t)$. Then, using (1.3.1), we have

$$\frac{d}{dt}\int_\alpha^\beta u dx = \frac{d}{dt}[\int_\alpha^{\xi^-(t)} u dx + \int_{\xi^+(t)}^\beta u dx] = -f(u)|_\alpha^\beta = -f(u(\beta,t)) + f(u(\alpha,t)).$$

or, differentiating the integrals

$$\int_\alpha^{\xi^-} u_t dx + u^-\dot{\xi}^- + \int_{\xi^+}^\beta u_t dx - u^+\dot{\xi}^+ = -f(u)|_\alpha^\beta.$$

The solution on either side of the discontinuity was assumed to be smooth, so (1.3.8) holds in (α, ξ^-) and (ξ^+, β) and can be used to replace the integrals. Additionally, since ξ is smooth, $\dot{\xi}^- = \dot{\xi}^+ = \dot{\xi}$. Thus, we have

$$f(u(\xi^-)) = f(u^-)$$
$$f(u(\xi^+)) = f(u^+)$$

$$-f(u)|_\alpha^{\xi^-} + u^-\dot{\xi} - f(u)|_{\xi^+}^\beta - u^+\dot{\xi} = -f(u)|_\alpha^\beta,$$

or

$$-f(u(\xi,t)) + \cancel{f(u(\alpha,t))} + u^-\dot{\xi} - \cancel{f(u(\beta,t))} + f(u(\xi^+,t)) - u^+\dot{\xi} = -\cancel{f(u(\beta,t))} + \cancel{f(u(\alpha,t))}$$

$$\dot{\xi}(u^+ - u^-) = f(u^+) - f(u^-). \tag{1.3.20}$$

Let

$$[q] \equiv q^+ - q^- \tag{1.3.21}$$

denote the jump in a quantity q and write (1.3.20) as

$$\boxed{[u]\dot{\xi} = [f(u)].} \tag{1.3.22}$$

Equation (1.3.22) is called the *Rankine-Hugoniot jump condition* and the discontinuity is called a *shock wave*. We can use the Rankine-Hugoniot condition to find a discontinuous solution of Example 1.3.4.

Example 1.3.5. For $t < 1$, the discontinuous solution of (1.3.16, 1.3.18) is as given in Example 1.3.4. For $t \geq 1$, we hypothesize the existence of a single shock wave, passing

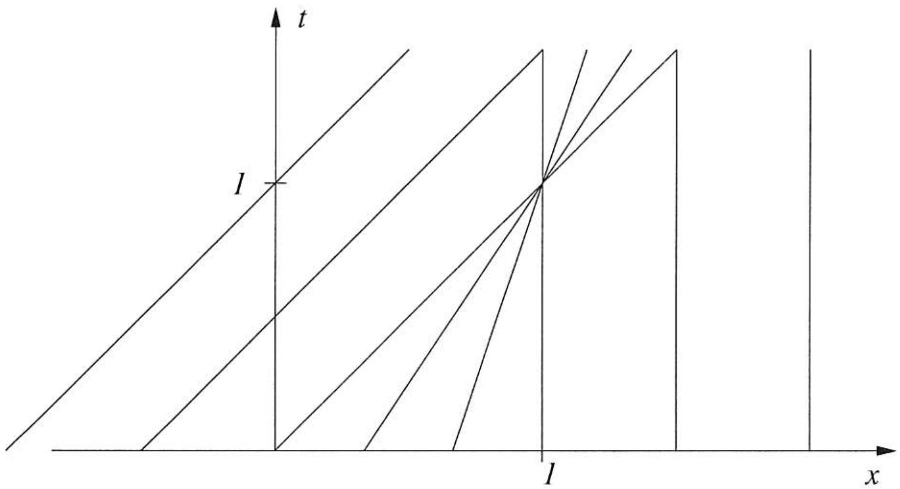


Figure 1.3.3: Characteristics for Burgers' equation (1.3.16) with initial data given by (1.3.18).

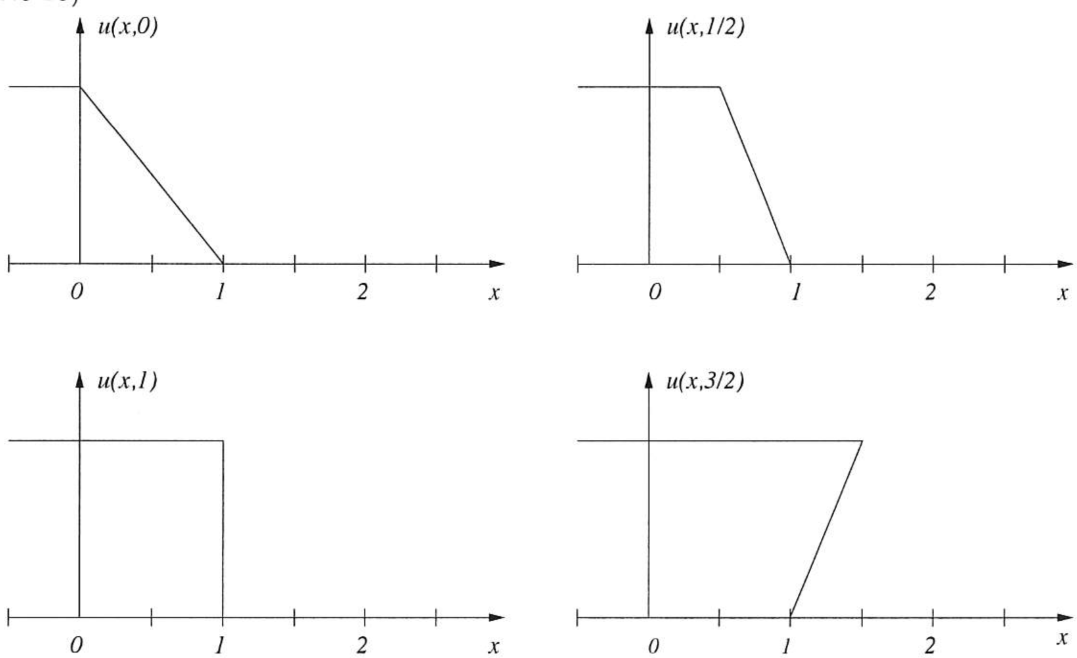


Figure 1.3.4: Multivalued solution of Burgers' equation (1.3.16) with initial data given by (1.3.18). The solution $u(x,t)$ is shown as a function of x for $t = 0$, 1/2, 1, and 3/2.

through $(1, 1)$ in the (x, t)-plane. As shown in Figure 1.3.5, the solution of Example 1.3.4 can be used to infer that $u^- = 1$ and $u^+ = 0$. Thus, $f(u^-) = (u^-)^2/2 = 1/2$ and $f(u^+) = (u^+)^2/2 = 0$. Using (1.3.22), the velocity of the shock wave is

$$\dot{\xi} = \frac{1}{2}.$$

$u^- = 1$

$u^+ = 0.$

Integrating, we find the shock location as

$$\xi = \frac{1}{2}t + c.$$

Since the shock passes through $(1, 1)$, the constant of integration $c = 1/2$, and

$$\xi = \frac{1}{2}(t + 1). \tag{1.3.23}$$

The characteristics and shock wave are shown in Figure 1.3.5 and the solution $u(x, t)$ is shown as a function of x for several times in Figure 1.3.6.

Let us consider another problem for Burgers' equation with different initial conditions that will illustrate another structure that arises in the solution of nonlinear hyperbolic systems.

Example 1.3.6. Consider Burgers' equation (1.3.16) subject to the initial conditions

$$\phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \le x < 1 \\ 1, & \text{if } 1 \le x \end{cases}. \tag{1.3.24}$$

$x - \phi(x_0)t = x_0$

Using (1.3.17) and (1.3.24), we see that the characteristic passing through $(x_0, 0)$ satisfies

$$x = \begin{cases} x_0, & \text{if } x_0 < 0 \\ x_0(1 + t), & \text{if } 0 \le x_0 < 1 \\ x_0 + t, & \text{if } 1 \le x_0 \end{cases}. \tag{1.3.25}$$

These characteristics, shown in Figure 1.3.7, may be used to verify that the solution, shown in Figure 1.3.8, is continuous. Additional considerations and difficulties with nonlinear hyperbolic systems are discussed in Lax [4].

Thus far, we have only considered initial value problems for (1.3.8). Initial-boundary problems are also possible; however, boundary conditions cannot be prescribed arbitrarily

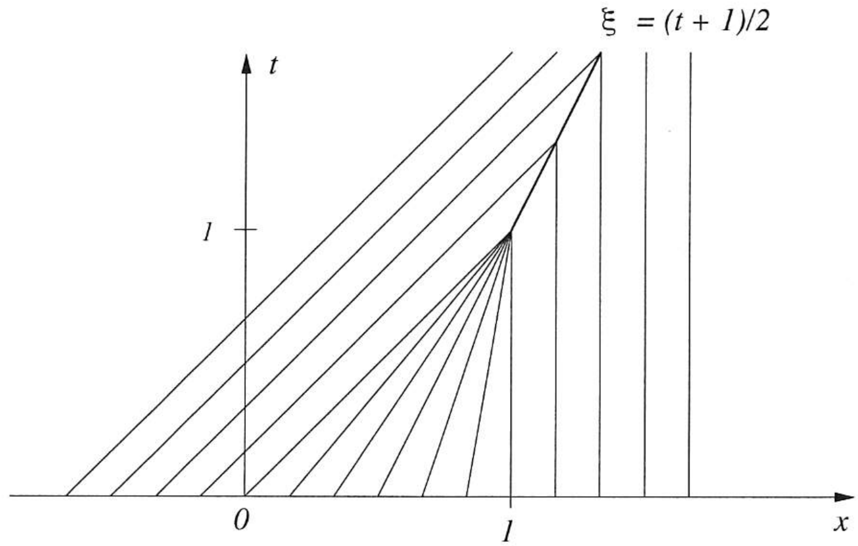


Figure 1.3.5: Characteristics and shock discontinuity for Example 1.3.5.

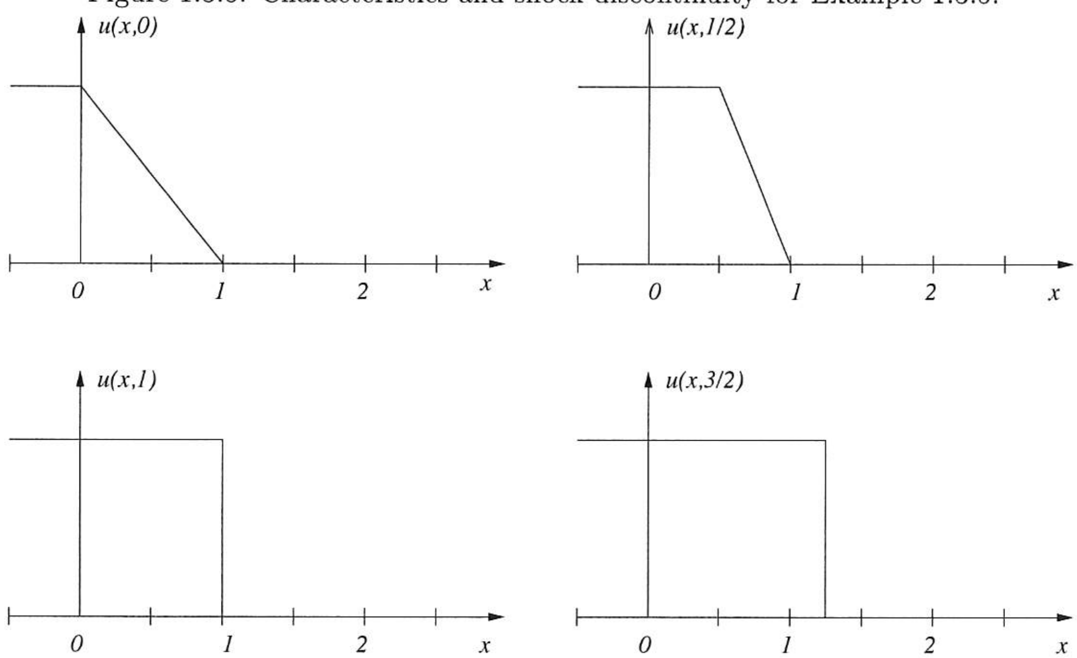


Figure 1.3.6: Solution $u(x,t)$ of Example 1.3.5 as a function of x at $t = 0$, $1/2$, 1, and $3/2$. The solution is discontinuous for $t > 1$.

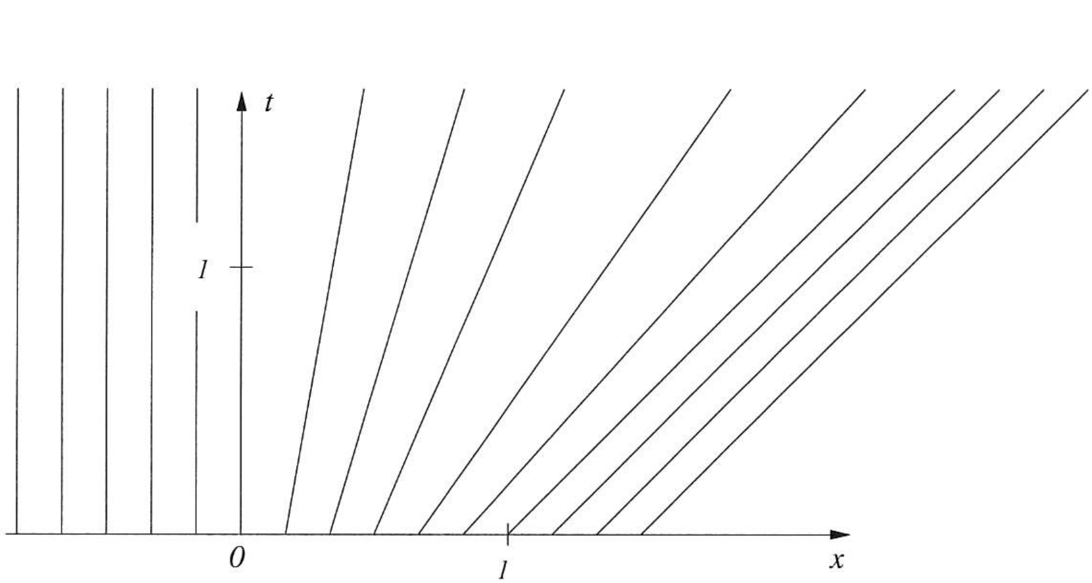


Figure 1.3.7: Characteristics for Example 1.3.6.

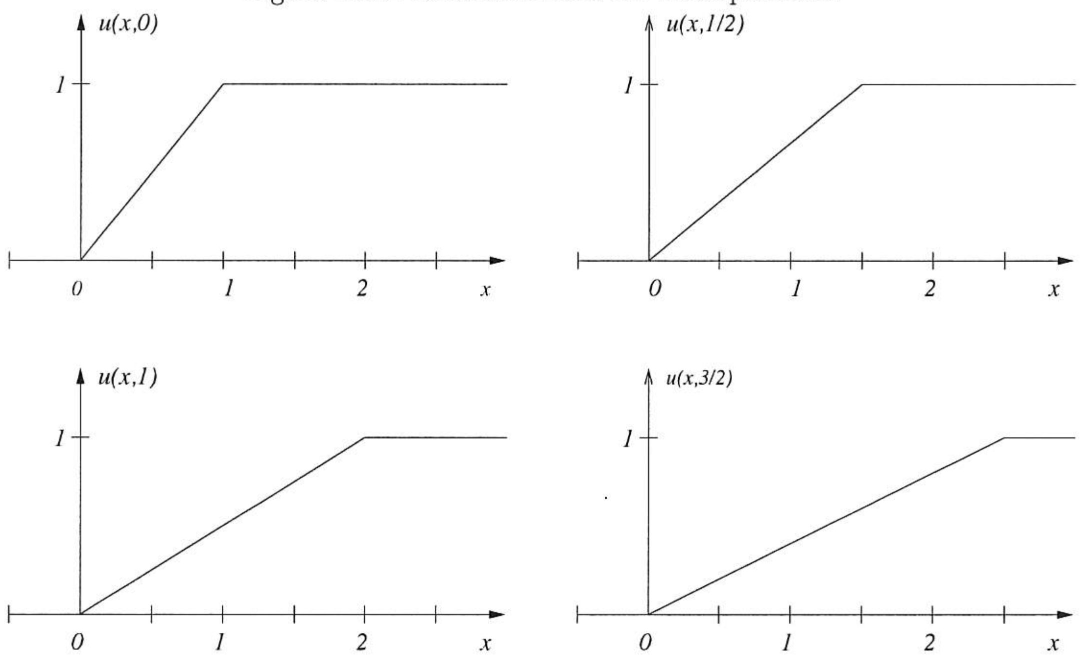


Figure 1.3.8: Solution $u(x,t)$ of Example 1.3.6 as a function of x at $t = 0$, $1/2$, 1, and $3/2$.

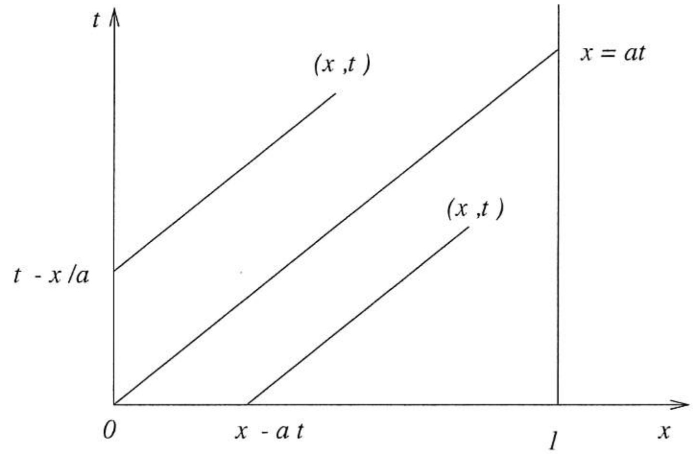


Figure 1.3.9: Characteristics for the linear initial-boundary value problem (1.3.26a).

and are determined by the characteristic directions (1.3.11). Let us begin with the linear problem

$$u_t + au_x = 0, \qquad 0 < x < 1, \qquad t > 0, \tag{1.3.26a}$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le 1. \tag{1.3.26b}$$

Suppose that $a > 0$. As shown in Figure 1.3.9, the solution at a point (x_1, t_1) where $x_1 > at_1$ is determined by the initial condition at $(x_1 - at_1, 0)$. On the other hand, the solution at a point (x_2, t_2) where $x_2 < at_2$ is not determined by the initial condition and requires a boundary value to be prescribed at $(0, t_2 - x_2/a)$. Therefore, the proper boundary condition to apply is

$$u(0,t) = \psi(t), \qquad t > 0. \tag{1.3.26c}$$

The solution of the initial-boundary value problem (1.3.26a) is given by

$$u(x,t) = \begin{cases} \psi(t - x/a), & \text{if } 0 < x < at \\ \phi(x - at), & \text{if } at \le x < 1 \end{cases}. \tag{1.3.27}$$

There are no boundary conditions at $x = 1$. However, if $a < 0$, the situation would be reversed and boundary conditions would be needed at $x = 1$ and not at $x = 0$.

Determining the proper boundary conditions for nonlinear problems can be quite complicated. For example, consider solving (1.3.9b) on $0 < x < 1$, $t > 0$ with $a(u(0,t)) >$

0 and $a(u(1,t)) < 0$. As shown on the left of Figure 1.3.10, such a problem would need boundary conditions at both $x = 0$ and $x = 1$. On the other hand, a problem with $a(u(0,t)) < 0$ and $a(u(1,t)) > 0$ (Figure 1.3.10, right) would not require any boundary conditions.

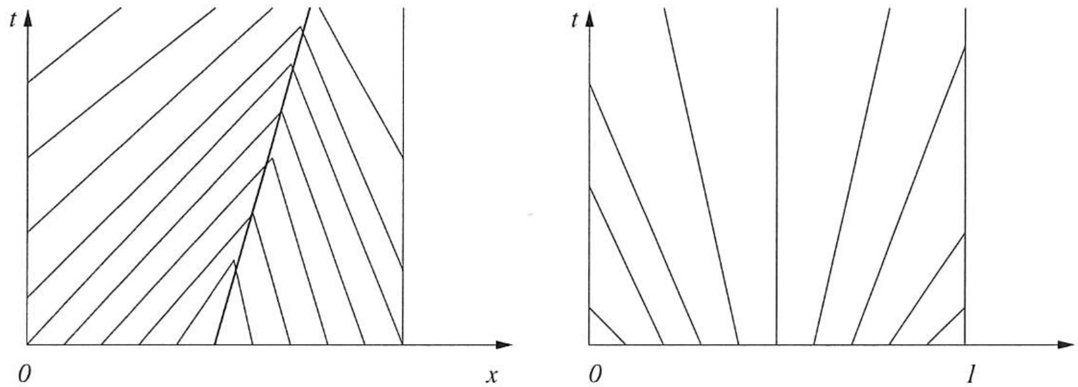


Figure 1.3.10: Characteristics for a nonlinear initial-boundary value problem with $a(u(0,t)) > 0$ and $a(u(1,t)) < 0$ (left) and with $a(u(0,t)) < 0$ and $a(u(1,t)) > 0$ (right).

Problems

1. A *Riemann problem* is a Cauchy (initial value) problem with piecewise constant initial data. Consider the Riemann problem for the inviscid Burgers' equation

$$u_t + \frac{1}{2}(u^2)_x = 0$$

$$u(x,0) = \begin{cases} u_L, & \text{if } x < 0 \\ u_R, & \text{if } x \geq 0 \end{cases} .$$

where u_L and u_R are constants. Solve this problem in the two cases when $u_L > u_R$ and when $u_L < u_R$. Sketch several characteristic curves and sketch the solution as a function of x at a few times. As a hint, you may want to first solve a problem with continuous initial data, *e.g.*,

$$u(x,0) = \begin{cases} u_L, & \text{if } x < -\epsilon \\ \frac{u_L}{2\epsilon}(\epsilon - x) + \frac{u_R}{2\epsilon}(\epsilon + x), & \text{if } -\epsilon < x \leq \epsilon \\ u_R, & \text{if } \epsilon < x \end{cases} ,$$

and take the limit as $\epsilon \to 0$. You may also consult Lax [4].

2. Use the method of characteristics to construct a solution of the inhomogeneous initial value problem

$$u_t + au_x = b(x), \qquad -\infty < x < \infty, \qquad t > 0,$$

$$u(x,0) = \phi(x), \qquad -\infty < x < \infty,$$

where a is a constant and $b(x)$ is smooth.

3. A scalar conservation law having cylindrical symmetry has the form

$$\frac{d}{dt}\int_\alpha^\beta ur\,dr = -f(u)r|_\alpha^\beta,$$

where r is a radial coordinate. If solutions are smooth this may be written as the first-order hyperbolic partial differential equation

$$u_t + \frac{1}{r}[f(u)r]_r = 0.$$

Solve an initial-boundary value problem for this equation on $r > 0$, $t > 0$ when $f(u) = au$ with a a positive constant. Assume the initial conditions prescribe

$$u(r,0) = \phi(r), \qquad r > 0,$$

where $\phi(r)$ is smooth. What boundary conditions, if any, must be prescribed?

4. Consider a linear vector systems of m conservation laws of the form

$$\mathbf{u}_t + \mathbf{A}\mathbf{u}_x = 0$$

where $\mathbf{A}$ is a constant $m \times m$ matrix. Let $\mathbf{P}$ be the $m \times m$ matrix that reduces $\mathbf{A}$ to the canonical form

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda}.$$

If $\mathbf{A}$ has m distinct real eigenvalues then $\mathbf{\Lambda}$ is the diagonal matrix

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{bmatrix}$$

and the partial differential system is called *hyperbolic*. (Additionally, if $\mathbf{\Lambda}$ has m complex eigenvalues the system is called *elliptic*. These extend the notions of hyperbolicity and ellipticity that were introduced in Section 1.2.

4.1. Assume that the given partial differential system is hyperbolic and introduce the transformation

$$\mathbf{u} = \mathbf{P}\mathbf{v}.$$

Show that $\mathbf{v}$ satisfies

$$\mathbf{v}_t + \mathbf{\Lambda}\mathbf{v}_x = \mathbf{0}.$$

Thus, the transformation has uncoupled the system (except possibly for the initial and/or boundary conditions). It now has the scalar form

$$(v_i)_t + \lambda_i (v_i)_x = 0, \qquad i = 1, 2, \cdots, m.$$

This system may be analyzed using characteristics in the same manner as for the scalar case.

4.2. Consider the wave equation written as a first-order system of two equations with

$$\mathbf{A} = \begin{bmatrix} 0 & -c \\ -c & 0 \end{bmatrix}.$$

Find $\mathbf{P}$, $\mathbf{\Lambda}$, and the diagonal system satisfied by $\mathbf{v}$. Determine the characteristics and, hence, $\mathbf{v}$. Knowing $\mathbf{v}$, determine $\mathbf{u}(x,t)$ such that it satisfies the initial data

$$\mathbf{u}(x,0) = \begin{bmatrix} u_1^0(x) \\ u_2^0(x) \end{bmatrix}.$$